All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $\{x_n\}$ be a sequence of real numbers.
   (i) Define the statement "$\{x_n\}$ is a Cauchy sequence".
   (ii) State Cauchy's General Principle of Convergence for $\{x_n\}$.

   (b) Define the statement "the function $f : (-\infty, 0) \to \mathbb{R}$ is uniformly continuous".

   (c) Let $f : (-\infty, 0) \to \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}$ be a sequence of real numbers such that $x_n < 0$, $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = 0$. Prove that the sequence $\{f(x_n)\}$ converges.
   [*Hint:* you may find it helpful to prove that $\{f(x_n)\}$ is a Cauchy sequence.]

2. Let $f : [a, b] \to \mathbb{R}$ be bounded.

   (a) Define the lower $L(f, P)$ and upper $U(f, P)$ Darboux sums of f with respect to a given partition P of the interval $[a, b]$.

   (b) Let the partition P' be a refinement of the partition P with one extra point. Prove that $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

   (c) Let the partition P' be any refinement of the partition P. Prove that $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.

   (d) Prove that $L(f, P) \leq U(f, Q)$ for any pair of partitions P and Q.

   (e) Define the lower Riemann integral $\underline{\int_a^b} f(x)\,dx$ and the upper Riemann integral $\overline{\int_a^b} f(x)\,dx$.

   (f) Prove that $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$.

   (g) Define what it means for f to be *Riemann integrable* on $[a, b]$.

3. In this question $I \subset \mathbb{R}$ is an interval with endpoints c and d. These endpoints may be real numbers or $\pm\infty$.

   (a) Define what it means for a function $F : I \to \mathbb{R}$ to be a *primitive* of the function $f : I \to \mathbb{R}$.

   (b) Suppose that the function $f : I \to \mathbb{R}$ has a primitive $F : I \to \mathbb{R}$. Is this primitive unique?

   (c) State the Fundamental Theorem of Calculus.

   (d) Define what it means for a function $F : I \to \mathbb{R}$ to be an *indefinite integral* of the locally Riemann integrable function $f : I \to \mathbb{R}$.

   (e) Suppose that the interval I is open (i.e. the endpoints do not belong to the interval) and that the function $f : I \to \mathbb{R}$ is continuous. Prove that any indefinite integral of f is a primitive of f.

   (f) Find the derivative of the function $G(x) = \displaystyle\int_{\sin x}^{\sin(x^2)} \sin(t^3)\, dt\,.$

4. (a) Suppose that the functions $f, g : (-1,1) \to \mathbb{R}$ are differentiable and that $f'(x) = g'(x)$ for all $x \in (-1,1)$. Prove that $f(x) = g(x) + c$ for all $x \in (-1,1)$, where c is a constant.

   (b) Define the notion of the *radius of convergence* of a real power series.

   (c) State the theorem about the differentiability of a real power series.

   (d) Find the radius of convergence of the power series $g(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k\,.$

   (e) Consider the function $g : (-1,1) \to \mathbb{R}$, $g(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k\,.$ Prove that

$$g'(x) = \frac{1}{1+x}.$$

   (f) Prove that $\ln(1+x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ for all $x \in (-1,1)$.

   [*Hint:* you may find it helpful to use the results from (a) and (e).]

5. (a) State Cauchy's Generalisation of the Mean Value Theorem.

(b) Let n be a nonnegative integer, let $a < 0 < b$ and let $f : (a, b) \to \mathbb{R}$ be $n+1$ times differentiable. Put

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}\,x^{n-1} + \frac{f^{(n)}(0)}{n!}\,x^n,$$

$$R_n(x) = f(x) - P_n(x).$$

Given an $x \in (0, b)$, prove that $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}\,x^{n+1}$ for some $\xi \in (0, x)$.

(c) Let $x > 0$. Prove that $\arctan x = x + \dfrac{3\xi^2 - 1}{3(1+\xi^2)^3}\,x^3$ for some $\xi \in (0, x)$.

(d) Prove that $\dfrac{16}{9}\sqrt{3} < \pi < 2\sqrt{3}$.

[*Hint:* apply the result from (c) with a particular x.]

6. In this question $I \subset \mathbb{R}$ is an interval with endpoints c and d. These endpoints may be real numbers or $\pm\infty$.

(a) Define what it means for a function $f : I \to \mathbb{R}$ to be *locally Riemann integrable.*

(b) Let $f : I \to \mathbb{R}$ be locally Riemann integrable. Define what it means for f to be integrable over I in the improper sense.

(c) State and prove the Comparison Theorem for Improper Integrals.

You may use without proof the following fact. Let $f : I \to [0, +\infty)$ be a locally Riemann integrable function. Then f is integrable over I if and only if there exists a constant K such that for any $a, b \in I$, $a < b$, we have $\int_a^b f(x)\,dx \le K$.

(d) Suppose that f is locally Riemann integrable over I and suppose that $\int_c^d |f(x)|\,dx$ exists. Prove that $\int_c^d f(x)\,dx$ exists.

(e) Prove the existence of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{x + x^2}\,dx$.

[*Hint 1:* examine whether the integrand has a limit as $x \to 0^+$.]

[*Hint 2:* you may find it helpful to use the results from (c) and (d).]

END OF PAPER